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A GENERALIZATION OF A LEMMA OF ABHYANKAR AND MOH

Stuart Sui-Sheng WANG*
Dept. of Mathematical Sciences, Oakland University, Rochester, MI 48063, USA
and Dept. of Mathematics, Purdue University, West Lafayette, IN 47907, USA

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In a fundamental paper [1], Abhyankar and Moh proved three important theorems: the Epimorphism Theorem, Automorphism Theorem and Embedding Theorem. Their proof made use of the following lemma [1, (2.5) Corollary, p. 256]: Let k be a ring and let $\alpha : k[X_1,\dots,X_p]\to k[Z]$ be a k-algebra homomorphism between two polynomial rings over k. Assume that α is surjective and $\operatorname{Ker}\alpha$ can be generated by $p-1$ elements, and let $f_2,\dots,f_p$ be any such elements. Then

$$\alpha\left(\frac{\partial(f_2,\dots,f_p)}{\partial(X_1,\dots,X_p;\mu)}\right)k[Z]=\frac{\mathrm{d}\alpha(X_\mu)}{\mathrm{d}Z}\,k[Z]\quad\text{for } 1\le\mu\le p.$$

Here

$$\frac{\partial(f_2,\dots,f_p)}{\partial(X_1,\dots,X_p;\mu)}$$

denotes the determinant of the Jacobian matrix

$$\left[\frac{\partial f_l}{\partial X_j}\right]_{j=1,\dots,\mu-1,\mu+1,\dots,p}^{l=2,\dots,p}.$$

They gave two proofs of this lemma, one in high school algebra [1, pp. 152–155], one in college algebra [1, pp. 156–158]. In this short note, we generalize this lemma and use junior-high algebra to prove it. We let

$$\frac{\partial(f_{q+1},\dots,f_p)}{\partial(X_1,\dots,X_p;\mu_1,\dots,\mu_q)}$$

denote the determinant of the matrix obtained from

$$\left[\frac{\partial f_l}{\partial X_j}\right]_{j=1,\dots,p}^{l=q+1,\dots,p}$$

by deleting the columns $\mu_1,\dots,\mu_q$.

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Theorem. *Let k be a commutative ring with identity, let $\alpha: k[X_1, \ldots, X_p] \to k[Z_1, \ldots, Z_q]$ be a k-algebra homomorphism between two polynomial rings over k, and let h_i be the image of X_i under α for $i = 1, \ldots, p$. Assume that α is surjective (so that $p \geq q$) and the kernel of α can be generated by $p-q$ elements, say,*

$$\operatorname{Ker} \alpha = (f_{q+1}, \ldots, f_p).$$

Then

$$\alpha\left(\frac{\partial(f_{q+1}, \ldots, f_p)}{\partial(X_1, \ldots, X_p; \mu_1, \ldots, \mu_q)}\right) k[Z_1, \ldots, Z_q] = \frac{\partial(h_{\mu_1}, \ldots, h_{\mu_q})}{\partial(Z_1, \ldots, Z_q)} k[Z_1, \ldots, Z_q]$$

for $1 \leq \mu_1 < \mu_2 < \cdots < \mu_q \leq p$.

Proof. Since α is surjective, there exist $t_1, \ldots, t_q \in k[X_1, \ldots, X_p]$ such that $t_1 \overset{\alpha}{\mapsto} Z_1, \ldots, t_q \overset{\alpha}{\mapsto} Z_q$. Then $X_i - h_i(t_1, \ldots, t_q) \overset{\alpha}{\mapsto} 0$ for $i = 1, \ldots, p$ and hence

$$X_i - h_i(t_1, \ldots, t_q) = \sum_{l=q+1}^{p} r_{i,l} f_l$$

for some $r_{i,l} \in k[X_1, \ldots, X_p]$. Applying $\partial/\partial X_j$ for $j = 1, \ldots, p$, we have

$$\delta_{ij} = \frac{\partial h_i(t_1, \ldots, t_q)}{\partial X_j} + \sum_l \frac{\partial r_{i,l}}{\partial X_j} f_l + \sum_l r_{i,l} \frac{\partial f_l}{\partial X_j}$$

where δ_{ij} is the Kronecker delta. Applying α, we have

$$\delta_{ij} = \alpha\left(\frac{\partial h_i(t_1, \ldots, t_q)}{\partial X_j}\right) + \sum_l \alpha(r_{i,l}) \alpha\left(\frac{\partial f_l}{\partial X_j}\right).$$

In matrix notation, we have

$$\begin{aligned} \mathbf{I}_{p \times p} = {} & \left[\alpha\left(\frac{\partial h_i(t_1, \ldots, t_q)}{\partial X_j}\right)\right]_{j=1, \ldots, p}^{i=1, \ldots, p} \\ & + [\alpha(r_{i,l})]_{l=q+1, \ldots, p}^{i=1, \ldots, p} \times \left[\alpha\left(\frac{\partial f_l}{\partial X_j}\right)\right]_{j=1, \ldots, p}^{l=q+1, \ldots, p}, \end{aligned}$$

where $\mathbf{I}_{p \times p}$ denotes the p-by-p identity matrix.

The chain rule states that

$$\left[\alpha\left(\frac{\partial h_i(t_1, \ldots, t_q)}{\partial X_j}\right)\right]_{j=1, \ldots, p}^{i=1, \ldots, p} = \left[\frac{\partial h_i(Z_1, \ldots, Z_q)}{\partial Z_k}\right]_{k=1, \ldots, q}^{i=1, \ldots, p} \times \left[\alpha\left(\frac{\partial t_k}{\partial X_j}\right)\right]_{j=1, \ldots, p}^{k=1, \ldots, q}.$$

Therefore, if we let

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial h_1}{\partial Z_1} \cdots \dfrac{\partial h_1}{\partial Z_q} & \alpha(r_{1,q+1}) \cdots \alpha(r_{1,p}) \\ \cdot \quad \cdot & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdot \quad \cdot \\ \dfrac{\partial h_p}{\partial Z_1} \cdots \dfrac{\partial h_p}{\partial Z_q} & \alpha(r_{p,q+1}) \cdots \alpha(r_{p,p}) \end{bmatrix}$$

and

$$\mathbf{J} = \begin{bmatrix} \alpha\left(\dfrac{\partial t_1}{\partial X_1}\right) & \cdots & \alpha\left(\dfrac{\partial t_1}{\partial X_p}\right) \\ \vdots & & \vdots \\ \alpha\left(\dfrac{\partial t_q}{\partial X_1}\right) & \cdots & \alpha\left(\dfrac{\partial t_q}{\partial X_p}\right) \\ \alpha\left(\dfrac{\partial f_{q+1}}{\partial X_1}\right) & \cdots & \alpha\left(\dfrac{\partial f_{q+1}}{\partial X_p}\right) \\ \vdots & & \vdots \\ \alpha\left(\dfrac{\partial f_p}{\partial X_1}\right) & \cdots & \alpha\left(\dfrac{\partial f_p}{\partial X_p}\right) \end{bmatrix}$$

then

$$(*) \qquad \mathbf{I}_{p\times p} = \mathbf{HJ}.$$

Taking the qth compound (i.e., the qth exterior power) leads to

$$\mathbf{I}_{\binom{p}{q}\times\binom{p}{q}} = \mathbf{H}^{(q)}\mathbf{J}^{(q)}$$

where $\binom{p}{q} = p!/q!(p-q)!$. Multiplying on the right by the qth adjugate compound of $\mathbf{J}$ it follows that

$$\mathbf{I}_{\binom{p}{q}\times\binom{p}{q}}(\operatorname{adj}^{(q)}\mathbf{J}) = \mathbf{H}^{(q)}\mathbf{J}^{(q)}(\operatorname{adj}^{(q)}\mathbf{J}),$$

i.e.,

$$\operatorname{adj}^{(q)}\mathbf{J} = \mathbf{H}^{(q)}(\det \mathbf{J})\mathbf{I}_{\binom{p}{q}\times\binom{p}{q}},$$

i.e.,

$$\operatorname{adj}^{(q)}\mathbf{J} = (\det \mathbf{J})\mathbf{H}^{(q)}.$$

Consequently,

$$(**) \qquad \alpha\left(\frac{\partial(f_{q+1},\dots,f_p)}{\partial(X_1,\dots,X_p;\mu_1,\dots,\mu_q)}\right) = (\det \mathbf{J})\,\frac{\partial(h_{\mu_1},\dots,h_{\mu_q})}{\partial(Z_1,\dots,Z_q)}.$$

However, $(*)$ implies that $1 = (\det \mathbf{H})(\det \mathbf{J})$ which shows that $\det \mathbf{J}$ is invertible in $k[Z_1,\dots,Z_q]$. This together with $(**)$ implies that

$$\alpha\left(\frac{\partial(f_{q+1},\dots,f_p}{\partial(X_1,\dots,X_p;\mu_1,\dots,\mu_q)}\right) \quad \text{and} \quad \frac{\partial(h_{\mu_1},\dots,h_{\mu_q})}{\partial(Z_1,\dots,Z_q)}$$

generate the same ideal in $k[Z_1,\dots,Z_q]$.

References

[1] S.S. Abhyankar and T.T. Moh, Embeddings of the line in the plane, J. reine angew. Math. 276 (1975) 148–166. MR 52 #407. Zbl. 332. 14004.

[2] A.C. Aitken, Determinants and Matrices, 9th edition (Interscience, New York, 1956). MR 1, 35.